Lecture 9

7.2: Trigonometric Integrals

In today's lecture, we will cover 3 types of integrals:

1. $\int \sin^m x \cos^n x \, dx$

2. $\int \sec^m x \tan^n x \, dx$

3. (i) $\int \sin(mx)\cos(nx)\, dx$

   (ii) $\int \sin(mx)\sin(nx)\, dx$

   (iii) $\int \cos(mx)\cos(nx)\, dx$

1. This splits into three cases

n odd If n is odd, we can write $n=2p+1$, then

$$\sin^m x \cos^n x = \sin^m x \cos^{2p+1} x = \sin^m x (\cos^2 x)^p \cos x$$

$$= \sin^m x (1-\sin^2 x)^p \cos x$$

So, the integral

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x (1-\sin^2 x)^p \cos x \, dx$$

can be done with a u-substitution of

$u = \sin x \qquad\qquad du = \cos x \, dx$

<u>Ex</u>: Compute $\int \sin^2 x \cos^3 x \, dx$

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x (1-\sin^2 x) \cos x \, dx$$

$(u = \sin x)$

$$= \int u^2 (1-u^2) \, du$$

$$= \int (u^2 - u^4) \, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= \boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 + C}$$

<u>m odd</u> This works just like the last case, but we instead use $\sin^2 x = 1 - \cos^2 x$, and the u-sub

$u = \cos x \qquad\qquad du = -\sin x \, dx$

<u>Ex</u>: Compute $\int \sin^5 x \cos^3 x \, dx$

$$\int \sin^5 x \cos^3 x \, dx = \int \sin x \sin^4 x \cos^3 x \, dx$$

$$= \int \sin x (1-\cos^2 x)^2 \cos^3 x \, dx \quad (u = \cos x)$$

$$= \int -(1-u^2)^2 u^3 \, du = -\int (u^3 - 2u^5 + u^7) \, du = -\frac{1}{4}u^4 + \frac{1}{3}u^6 - \frac{1}{8}u^8 + C$$

$$= \boxed{-\frac{1}{4}\cos^4 x + \frac{1}{3}\cos^6 x - \frac{1}{8}\cos^8 x + C}$$

If both m&n are even, we use the identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \& \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

Sometimes, using

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

can be useful. Another alternative is to change everything to powers of sine or cosine using

$$\sin^2 x + \cos^2 x = 1$$

Ex: Compute $\int \sin^4 x \cos^2 x \, dx$

$$\int \sin^4 x \cos^2 x \, dx = \int \sin^2 x \left(\sin^2 x \cos^2 x\right) dx$$

$$= \int \frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}\sin 2x\right)^2 dx$$

$$= \frac{1}{8}\int \left(\sin^2 2x - \sin^2(2x)\cos(2x)\right) dx$$

$u = \sin 2x$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx - \frac{1}{8}\int \sin^2(2x)\cos(2x)\, dx$$

$$= \frac{1}{16}\left(x - \frac{1}{4}\sin 4x\right) - \frac{1}{16}\int u^2 du = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x - \frac{1}{3}u^3\right) + C$$

$$= \boxed{\frac{1}{16}\left(x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right) + C}$$

② There is again three subcases

m even & $(n \geq 1$ or $m \geq 4)$ (m>0) fine even if $m \geq 2$, but overpowered

We reserve one factor of $\sec^2 x$ and turn everything else into powers of $\tan x$ using

$$\sec^2 x = 1 + \tan^2 x$$

then we use the u-substitution

$$u = \tan x \qquad du = \sec^2 x \, dx$$

Ex: Compute $\int \tan^5 \theta \sec^4 \theta \, d\theta$

$$\int \tan^5\theta \sec^4\theta \, d\theta = \int \tan^5\theta \sec^2\theta \sec^2\theta \, d\theta$$

$$= \int \tan^5\theta (1+\tan^2\theta) \sec^2\theta \, d\theta \quad (u = \tan\theta)$$

$$= \int u^5(1+u^2)\, du = \int (u^5 + u^7)\, du = \frac{1}{6}u^6 + \frac{1}{8}u^8 + C = \boxed{\frac{1}{6}\tan^6\theta + \frac{1}{8}\tan^8\theta + C}$$

n odd & $m \geq 1$

In this case we reserve a factor of $\tan x \sec x$ and write everything else in terms of $\sec x$. (this is why we need $m \geq 1$ & taking one power of $\tan x$ leaves an even power of $\tan x$ left). We then use the u-sub:

$$u = \sec x \qquad du = \tan x \sec x \, dx$$

<u>Ex</u>: Compute $\int \sec^3 y \tan y \, dy$

$$\int \sec^3 y \tan y \, dy = \int \sec^2 y (\sec y \tan y) \, dy \quad (u = \sec y)$$

$$= \int u^2 \, du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\sec^3 y + C}$$

<u>m odd & n even</u> Here we turn all of the tangents into secants (which we can do since there are an even number of them). Then we use one of the formulas for integrals of powers of secant:

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$(n \geq 3) \quad \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$$

The other integral to know is

$$\int \tan x \, dx = \ln|\sec x| + C$$

<u>Ex</u>: Compute $\int \sec y \tan^2 y \, dy$

$$\int \sec y(\sec^2 y - 1)\, dy = \int (\sec^3 y - \sec y)\, dy$$

$$= \boxed{\frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C}$$

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③ To compute the integrals of this type we use the identities

① $\sin[(m-n)x] = \sin(mx)\cos(nx) - \cos(mx)\sin(nx)$

② $\sin[(m+n)x] = \sin(mx)\cos(nx) + \cos(mx)\sin(nx)$

③ $\cos[(m-n)x] = \cos(mx)\cos(nx) + \sin(mx)\sin(nx)$

④ $\cos[(m+n)x] = \cos(mx)\cos(nx) - \sin(mx)\sin(nx)$

to get the identities

$$\sin(mx)\cos(nx) = \frac{1}{2}\Big[\sin[(m-n)x] + \sin[(m+n)x]\Big] \qquad \frac{①+②}{2}$$

$$\sin(mx)\sin(nx) = \frac{1}{2}\Big[\cos[(m-n)x] - \cos[(m+n)x]\Big] \qquad \frac{③-④}{2}$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\Big[\cos[(m-n)x] + \cos[(m+n)x]\Big] \qquad \frac{③+④}{2}$$

Ex: Compute $\int \cos \pi x \cos 4\pi x \, dx$

$$\int \cos \pi x \cos 4\pi x \, dx = \frac{1}{2}\int \left[\cos 3\pi x + \cos 5\pi x\right] dx$$

$$= \frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x + \frac{1}{5\pi}\sin 5\pi x\right) + C$$

$$= \boxed{\frac{1}{6\pi}\sin 3\pi x + \frac{1}{10\pi}\sin 5\pi x + C}$$

Now, we work through integrals of powers of secant and tangent.

Ex: Compute $\int \sec x \, dx$ $\left(\begin{array}{l} u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x)\, dx \end{array}\right)$

$$\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{1}{u} \, du = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$$

(There is a much more straightforward way to do this integral, but it requires using partial fractions, which we cover soon.)

Ex: Compute $\int \tan x \, dx$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int -\frac{1}{u} \, du = -\ln|u| + C = -\ln|\cos x| + C$$

$(u = \cos x)$

$$= \boxed{\ln|\sec x| + C}$$

Ex: $\int \sec^2 x\,dx = \boxed{\tan x + C}$

Ex: $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \boxed{\tan x - x + C}$

Ex: $\int \sec^3 x\,dx$ $\quad \begin{pmatrix} u = \sec x & dv = \sec^2 x\,dx \\ du = \sec x \tan x\,dx, & v = \tan x \end{pmatrix}$

$= \sec x \tan x - \int \sec x \tan^2 x\,dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$

$= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx$

$= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx$

$\Rightarrow \boxed{\int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C}$

Ex: $(n \geq 2)$ $\int \tan^n x\,dx$ $\quad \left(\begin{matrix} \text{let } n = k+2 \ (k \geq 0) \\ k = n-2 \end{matrix}\right)$

$\int \tan^n x\,dx = \int \tan^k x \tan^2 x\,dx = \underbrace{\int \tan^k x \sec^2 x\,dx}_{u = \tan x} - \int \tan^k x\,dx$

$= \int u^k\,du - \int \tan^k x\,dx = \frac{1}{k+1}\tan^{k+1} x - \int \tan^k x\,dx$

$= \boxed{\frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx}$

Ex: $(n \geq 3)$ $\int \sec^n x \, dx$

Let $n = k+2$ $(k \geq 1)$ $\Rightarrow k = n-2$.

$$\int \sec^n x \, dx = \int \sec^k x \sec^2 x \, dx \quad \left| \begin{array}{ll} u = \sec^k x & dv = \sec^2 x \, dx \\ du = k \sec^{k-1} x \cdot \sec x \tan x \, dx & v = \tan x \end{array} \right.$$

$$= \sec^k x \tan x - k \int \sec^k x \tan^2 x \, dx$$

$$= \sec^k x \tan x - k \left(\int \sec^k x (\sec^2 x - 1) \, dx \right)$$

$$= \sec^k x \tan x - k \left(\int \sec^{k+2} x \, dx - \int \sec^k x \, dx \right)$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$\Rightarrow (n-1) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$$

$$\Rightarrow \boxed{\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx}$$